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Universal topological properties

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Universal topological properties

Until explicitly stated, all spaces in consideration are completely regular. Thus the abbreviation "space" means always "completely regular space".

Introduction. Let $\mathcal{P}$ be a property of topological spaces. We call $\mathcal{P}$ a universal property if every space X is homeomorphic with a dense subset of a space γX with property $\mathcal{P}$, such that each continuous map of X into any space Y satisfying $\mathcal{P}$, can be extended continuously to the whole of γX.

It turns out that the universal properties are precisely those properties, which are possessed by all compact spaces and which are inherited by closed subsets and (arbitrary) topological products.

§ 1. Almost-fitting properties, maximal embedding

Conventions. Let $\mathcal{P}$ be a property of topological spaces.
$\mathcal{P}$ is called productive or sometimes arbitrary productive, if the product of an arbitrary collection of spaces enjoying $\mathcal{P}$, has property $\mathcal{P}$.
$\mathcal{P}$ is called countably productive (respectively finitely productive), if the product of a countable (respectively finite) collection of spaces enjoying $\mathcal{P}$ has property $\mathcal{P}$.
$\mathcal{P}$ is called hereditary (respectively closed-hereditary) if every subspace (respectively closed subspace) of a space satisfying $\mathcal{P}$, has property $\mathcal{P}$.
$\mathcal{P}$ is called almost-fitting property, if whenever f is a perfect [1] map of a space X onto a space Y, then X has property $\mathcal{P}$ if Y has property $\mathcal{P}$.

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[1] A mapping f of a space X into a space Y will be called perfect if f is continuous, closed (the images of closed sets are closed) and the inverses of points are compact.

$\mathcal{P}$ is called a <u>fitting</u> property, if whenever f is a perfect map of a space X onto a space Y, then X has property $\mathcal{P}$ if and only if Y has property $\mathcal{P}$.

Compactness and realcompactness [1] are examples of properties which are closed-hereditary and productive. Both are also almost-fitting properties.
Local compactness, σ-compactness, countable compactness, paracompactness, countable paracompactness, Čech-completeness are examples of properties which are closed-hereditary (but not productive). Each of these listed properties is an almost-fitting property.

If a topological space X is densely embedded in a space γX with property $\mathcal{P}$ then we call γX a <u>$\mathcal{P}$-fication</u> of X.
Sometimes γX is of the type that to each continuous mapping f of X into any space Y with property $\mathcal{P}$, we can find a continuous extension $\tilde{f}$ of f which carries γX into Y. γX is then said to be a <u>maximal</u> $\mathcal{P}$-fication of X. It is easy to see that in the latter case γX is uniquely determined to X, and we have $\gamma X = X$ if and only if X has property $\mathcal{P}$.

We call $\mathcal{P}$ a <u>universal property</u> if every space has a maximal $\mathcal{P}$-fication.

Compactness and realcompactness are indisputably the most interesting universal properties. The maximal $\mathcal{P}$-fications are here respectively the Čech-Stone compactification and the Hewitt realcompactification. The following theorem which is the main result of this section shows that universal properties are most familiar to us.

<u>Main result of §1.</u>

If $\mathcal{P}$ is a property of topological spaces, then the following statements are equivalent.

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[1] For the definition of realcompactness cf. [1] .

(a) $\mathcal{P}$ is a universal property.

(b) $\mathcal{P}$ is closed-hereditary, productive, and each compact space has property $\mathcal{P}$.

Before we attack the proof, we give some preliminary results which are of interest in itself.

<u>(1.1) Lemma.</u> Let $\mathcal{P}$ be a topological property which is productive and closed-hereditary.
If Z is a space and $\{X_\alpha | \alpha \in A\}$ is a collection of subspaces with property $\mathcal{P}$ then $X = \cap\{X_\alpha | \alpha \in A\}$ satisfies property $\mathcal{P}$.
An analogous result is obtained for properties that are only countably or even finitely productive.

<u>Proof.</u> Let $Y = \prod\{X_\alpha | \alpha \in A\}$, and $\Delta \subset Y$ given by $\Delta = \{x = (x_\alpha) \in Y | x_{\alpha_1} = x_{\alpha_2}$ $\forall \alpha_1, \alpha_2 \in A\}$.
It is not hard to see that X is homeomorphic with the subspace Δ.
Thus it remains to show that Δ has property $\mathcal{P}$.
Y has property $\mathcal{P}$ since each X_α has property $\mathcal{P}$ and $\mathcal{P}$ is productive.
Δ is a closed subset of Y because each X_α is a Hausdorffspace. Hence Δ has property $\mathcal{P}$ since $\mathcal{P}$ is closed-hereditary.

<u>(1.2) Theorem.</u> If a property $\mathcal{P}$ of topological spaces is closed-hereditary, productive and an invariant for the taking of open subsets, then $\mathcal{P}$ is a hereditary property.
Indeed, if Y is a space having $\mathcal{P}$ and $X \subset Y$ then $X = \bigcap\{Y \setminus \{p\} | p \in Y \setminus X\}$ i.e. X is intersection of open subsets of Y. By assumption each open subset of Y has property $\mathcal{P}$ and the preceding lemma yields that every intersection of spaces enjoying $\mathcal{P}$ has $\mathcal{P}$. Consequently X has property $\mathcal{P}$.

This theorem can serve as a test to decide whether some property is inherited by open subsets, closed subsets or (arbitrary) topological products.

For instance, it is easy to see that the property k [1] is an invariant for the taking of open and closed subsets. Since the property k is not hereditary the above result shows that the property k is not productive.

<u>(1.3) Lemma.</u> Let f be a continuous mapping of a space X into a space Y and suppose that $Z \subset Y$. Then $f^{-1}(Z)$ is homeomorphic with a closed subspace of $X \times Z$.

<u>Proof.</u> We shall proof that the graph of $g = f|_{f^{-1}(Z)}$ (which is homeomorphic with $f^{-1}(Z)$) is closed in $X \times Z$. Let (x, z) be any point of $X \times Z$ which is not in the graph of g, We propose that $f(x) \neq z$. Indeed the assertion $f(x) = z$ implies that $x \in f^{-1}(z) \subset f^{-1}(Z)$ i.e. $f(x) = g(x) = z$, which is impossible since we have supposed that (x, z) is not a point of the graph of g.
We can choose disjoint neighborhoods U(f(x)) and U(z) of f(x) and z in Y respectively. The continuity of $f : X \to Y$ insures us the existence of an neighborhood V(x) of x in X which is mapped inside U(f(x)) by f. Now $V(x) \times (U(z) \cap Z)$ is a neighborhood of (x, z) in $X \times Z$ which is disjoint from the graph of g. Since (x, z) was arbitrarily chosen, we conclude that the graph of g is closed in $X \times Z$.

From (1.3) we derive the following two general results.

<u>(1.4) Theorem.</u> Let $\mathcal{P}$ be a property of topological spaces which is finitely productive and closed-hereditary. If f is a continuous mapping from a space X with property $\mathcal{P}$ into a space Y, then the total preimage of each subset of Y with property $\mathcal{P}$ satisfies again property $\mathcal{P}$.

<u>(1.5) Theorem.</u> Let $\mathcal{P}$ be a property of topological spaces which is closed-hereditary. If for any space Y with property $\mathcal{P}$, the product of Y with any compact space Z has property $\mathcal{P}$, then $\mathcal{P}$ is an almost-fitting property.

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[1] A space X has property k provided that a subset is closed if it has a compact intersection with each compact subspace of X.

Proof. Let Y be a space with property $\mathcal{P}$, and suppose that f is a perfect map of a space X onto Y; we must show that X has property $\mathcal{P}$.
Let $\tilde{f}$ be the extension of f which carries βX into βY. (βX and βY denoting the Čech-Stone compactifications of X and Y respectively).
A well known theorem of Henriksen and Isbell states that $\tilde{f}^{-1}(Y) = X$ (cf. [1]). Hence by (1.3) $\tilde{f}^{-1}(Y) = X$ is homemorphic with a closed subspace of $\beta X \times Y$. The theorem now follows from the assumptions we made on the property $\mathcal{P}$.

(1.6) Lemma. If ϕ is a continuous map of a space Y into a space Z, whose restriction to a dense set X is a homeomorfism, then ϕ carries $Y \setminus X$ into $Z \setminus \phi(X)$.

Proof. See for instance [2] blz. 92.

Proof of the main result.

(a) => (b). Let $\mathcal{P}$ be a universal property; for each space X set γX the maximal $\mathcal{P}$-fication of X.
If X is compact then obviously X is closed in γX i.e. $X = \gamma X$ has property $\mathcal{P}$. So it remains to show that $\mathcal{P}$ is productive and closed-hereditary.
Let $\{X_\alpha \mid \alpha \in A\}$ be a collection of spaces enjoying $\mathcal{P}$ and $X = \prod\{X_\alpha \mid \alpha \in A\}$.
Each projection map $\pi_\alpha : X \to X_\alpha$ has a continuous extension $\pi_\alpha^* : \gamma X \to X_\alpha$.
Let $i^* : \gamma X \to X$ be defined by the conditions $(i^*(x))_\alpha = \pi_\alpha^*(x)$ $(\alpha \in A)$.
i^* is the identity on X, so we have by (1.6) that $\gamma X \setminus X = \phi$ i.e. $\gamma X = X$.
Consequently X has property $\mathcal{P}$.
Let X be a closed subset of a space Y satisfying $\mathcal{P}$. The inclusion map of X into Y has a continuous extension i^* of γX into Y. By (1.6) the preimage of the closed set X under i^* is X; hence X is closed in γX i.e. $\gamma X = X$. It follows that X has property $\mathcal{P}$.

(b) => (a). Let $\mathcal{P}$ possess the already cited invariances; let X be a space and βX its Čech-Stone compactification.
Consider for each continuous mapping f which sends X onto a dense subset of a space Y satisfying $\mathcal{P}$, the extension $\tilde{f}$ of f which carries βX onto βY, and set $X(Y, f) = \tilde{f}^{-1}(Y)$.

It follows from theorem (1.4) that $X(Y, f)$ has property $\mathcal{P}$.
Now let $\gamma X = \bigcap\{X(Y, f) \mid Y$ satisfies $\mathcal{P}$; $f: X \to Y$ continuous; fX dense in $Y\}$.
X is clearly densely embedded in γX moreover it follows from (1.1) that γX has property $\mathcal{P}$.
We shall prove that γX is a maximal $\mathcal{P}$-fication. If g is any continuous mapping from X into a space Z satisfying $\mathcal{P}$, then let Z' be the closure of gX in Z. Z' satisfies $\mathcal{P}$ since $\mathcal{P}$ is closed-hereditary.
Now we have $\gamma X \subset X(Z', g)$ (g considered a mapping of X into Z') and $\tilde{g}|\gamma X : \gamma X \to Z' \subset Z$ is a continuous extension of g which carries γX into Z.

§2. Examples of universal properties

We will show that there are "enough" universal properties (the theory above would obviously be not succesful if compactness and realcompactness were the only candidates).

Definition. A family of subsets of a topological space X has the $\underline{m}$-intersection property ($\underline{m}$ finite or infinite cardinal number) provided that every subcollection of cardinal $\leq \underline{m}$ has a nonempty intersection.
An ultrafilter $\mathcal{F}$ in X is said to be an $\underline{m}$-ultrafilter if the closed sets of X that are members of $\mathcal{F}$, satisfy the $\underline{m}$-intersection property.
A space X is called $\underline{m}$-ultracompact provided that every $\underline{m}$-ultrafilter in X is covergent.

Obviously compact implies $\underline{m}$-ultracompact for every $\underline{m}$; if $\underline{n} \leq \underline{m}$ then $\underline{n}$-ultracompact implies $\underline{m}$-ultracompact.
It is also easy to see that if X has the Lindelöfproperty then X is $\aleph_0$-ultracompact. The connection between $\aleph_0$-ultracompactness and realcompactness is considered on an other occasion.
(We can prove that $\aleph_0$-ultracompactness is equivalent to realcompactness for normal spaces).

(2.1) Lemma. Let $\mathcal{F}$ be an $\underline{\underline{m}}$-ultrafilter in a space X and $f : X \to Y$ a continuous mapping. The collection $\mathcal{G} = \{f(F)|F\in\mathcal{F}\}$ constitutes a base for an $\underline{\underline{m}}$-ultrafilter in Y.

Proof. A well known argument shows that $\mathcal{G}$ is base for an ultrafilter $\mathcal{G}'$ in X. Let $\{S_\alpha|\alpha\in A\}$ be a family of closed sets of $\mathcal{G}'$ with cardinal $\leq \underline{\underline{m}}$. Clearly every S_α intersects every $f(F)$ $(F\in\mathcal{F})$. Consequently every $f^{-1}(S_\alpha)$ $(\alpha\in A)$ is a closed subset of X and meets every member of $\mathcal{F}$. Hence, since $\mathcal{F}$ is an $\underline{\underline{m}}$-ultrafilter, $\{f^{-1}(S_\alpha)|\alpha\in A\}$ is a subcollection of $\mathcal{F}$ and $\cap\{f^{-1}(S_\alpha)|\alpha\in A\} \neq \phi$. It follows that $\{S_\alpha|\alpha\in A\}$ has non-empty intersection.

(2.2) Theorem. The property $\underline{\underline{m}}$-ultracompactness is closed-hereditary and productive for every $\underline{\underline{m}}$. Hence since every compact space is $\underline{\underline{m}}$-ultracompact, $\underline{\underline{m}}$-ultracompactness is a universal property.

Proof. Let $\{X_\alpha|\alpha\in A\}$ be a collection of $\underline{\underline{m}}$-ultracompact spaces and $X = \pi\{X_\alpha|\alpha\in A\}$. Take an $\underline{\underline{m}}$-ultrafilter $\mathcal{F}$ in X and let for $\alpha\in A$ $\mathcal{F}_\alpha = \{\pi_\alpha F|F\in\mathcal{F}\}$. By the previous lemma, each $\mathcal{F}_\alpha$ is base for an $\underline{\underline{m}}$-ultrafilter in X_α which is convergent to a point p_α in X_α.
Let p be the point of X whose α'th coordinate is p_α. A well known argument shows that p is limitpoint of $\mathcal{F}$, i.e. $\mathcal{F}$ is convergent (since $\mathcal{F}$ is an ultrafilter).
Now let X be an $\underline{\underline{m}}$-ultracompact space and Y a closed subspace of X.
We will show that Y is $\underline{\underline{m}}$-ultracompact.
Take an $\underline{\underline{m}}$-ultrafilter $\mathcal{F}$ in Y. The preceding lemma shows that $\mathcal{F}$ is base for an $\underline{\underline{m}}$-ultrafilter $\mathcal{F}'$ on X which is convergent, say to $p\in X$. Clearly $p\in\cap\{\bar{\bar{F}}|F\in\mathcal{F}'\}\subset\cap\{\bar{\bar{F}}|F\in\mathcal{F}\} = \cap\{\bar{\bar{F}}^Y|F\in\mathcal{F}\}$. Hence $\mathcal{F}$ is a convergent filter in Y.

(2.3) Theorem. Every space Y which is the perfect f-image of some $\underline{\underline{m}}$-ultracompact space X, is $\underline{\underline{m}}$-ultracompact. Hence together with (1.5) we conclude that $\underline{\underline{m}}$-ultracompactness is a fitting property.

Proof. Let $\mathcal{F}$ be an arbitrary $\underline{m}$-ultrafilter in Y and $\mathcal{G}$ an ultrafilter in X which contains the family $f^{-1}(\mathcal{F}) = \{f^{-1}(F) | F \in \mathcal{F}\}$.
We shall first prove that $\mathcal{G}$ is an $\underline{m}$-ultrafilter in X. Let us suppose that there exists a countable family $\mathcal{S}$ of closed members of $\mathcal{G}$ with empty intersection. Without lost of generality we may suppose that $\mathcal{S}$ is closed under finite intersections. The members of $f(\mathcal{S}) = \{f(S) | S \in \mathcal{S}\}$ are closed subsets of Y and they intersect each member of $\mathcal{F}$. Consequently $f(\mathcal{S}) \subset \mathcal{F}$ and we are able to choose $p \in \cap f(\mathcal{S})$ since $\mathcal{F}$ is an $\underline{m}$-ultrafilter in Y. Now $\{f^{-1}(p) \cap S | S \in \mathcal{S}\}$ is a centered system in X and compactness of $f^{-1}(p)$ yields $\{f^{-1}(p) \cap S | S \in \mathcal{S}\} \neq \phi$. Hence $\cap \mathcal{S} \neq \phi$, which is a contradiction.

The space X being $\underline{m}$-ultracompact, we have $\cap \overline{\mathcal{G}}^X \neq \phi$, and in consequence $\cap \overline{\mathcal{F}}^Y \neq \phi$.

(2.3) Lemma. If X is an $\underline{m}$-ultracompact space and if every open cover of X of cardinal $\leq \underline{m}$ has a finite subcover, then X is compact.

Proof. Let $\mathcal{F}$ be an arbitrary ultrafilter in X. Clearly the family of closed subsets of X that are members of $\mathcal{F}$ satisfy the $\underline{m}$-intersection property (otherwise their complements would constitute at least one open cover with cardinal $\leq \underline{m}$ that has no finite subcover).
$\underline{m}$-ultracompactness of X now yields that $\mathcal{F}$ is convergent. Consequently each ultrafilter in X is convergent i.e. X is compact.

In particular it follows that a topological space X is compact iff X is $\aleph_0$-ultracompact and countably compact. Actually a stronger result is true: X is compact <=> X is pseudocompact and realcompact.

(2.4) Theorem. For each (infinite) cardinal number $\underline{m}$ there exists a normal space X which is $\underline{m}$-ultracompact but not $\underline{n}$-ultracompact for $\underline{n} < \underline{m}$.

Proof. We may suppose $\underline{m} > \aleph_0$.
Let α be the smallest ordinal number of potency $\underline{m}$. Let $W = \{\xi \text{ ordinal} | \xi < \alpha\}$ and $W^* = \{\xi | \xi \leq \alpha\}$ be supplied with the usual order topology.

W is $\underline{m}$-ultracompact. For, since βW is homeomorphic to W^*, and ultrafilter $\mathfrak{F}$ in W that has no limit point in W must contain the $\underline{m}$ sets $F_\beta = \{\xi \in W | \xi \geq \beta\}$ $(\beta < \alpha)$. Since $\bigcap\{F_\beta | \beta < \alpha\} = \phi$, $\mathfrak{F}$ cannot be an $\underline{m}$-ultrafilter.

If $\underline{n} \leq \underline{m}$, then W is not $\underline{n}$-ultracompact. Indeed $\underline{n}$-ultracompactness would together with the fact that every open cover of W of cardinal $\leq n$ has a finite subcover, disprove (2.3).

References.

[1] L. Gillman and M. Jerison, Rings of continuous functions, van Nostrand 1960.

[2] M. Henriksen and J.R. Isbell, Some properties of compactifications, Duk. Math. Journ. Vol. 25, No. 1, pp. 83-106 (1958).